ON NEUMANN BOUNDARY PROBLEM FOR STRONGLY DEGENERATE PARABOLIC-HYPERBOLIC EQUATIONS ON A BOUNDED RECTANGLE

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ABSTRACT. We study a Neumann type initial-boundary value problem for strongly degenerate parabolic-hyperbolic equations under the nonlinearity-diffusivity condition. We suggest a notion of entropy solution for this problem and prove its uniqueness. The existence of entropy solutions is also discussed.

1. INTRODUCTION

In this paper, we consider the following initial-boundary value problem:

$$\begin{cases} u_t + \nabla \cdot \mathbf{f}(u) = \triangle B(u), \qquad (\mathbf{x},t) \in \Omega \times (0,T), \\ u|_{t=0} = u_0(\mathbf{x}), \qquad \mathbf{x} \in \Omega \\ (\mathbf{f}(u) - \nabla B(u)) \cdot \mathbf{n}(\mathbf{x},t) = 0, \qquad (\mathbf{x},t) \in \partial\Omega \times (0,T), \end{cases} \tag{1.1}$$

where Ω is an open and bounded rectangle of $\mathbb{R}^N$, $T > 0$ and $\mathbf{n}(\mathbf{x},t)$ denotes the outward unit normal vector to $\partial\Omega$ at $(\mathbf{x},t)$. $u_0(\mathbf{x}) : \Omega \to \mathbb{R}$ is a bounded function and $B(u)$ is a non-decreasing function in $\mathcal{C}^1(\mathbb{R})$. The flux function $\mathbf{f}(u)$ is assumed to be regular from $\mathbb{R}$ to $\mathbb{R}^N$.

Partial differential equations of the above form arise in mathematical models of many physical situation, such as two phase flows in porous media, sedimentation-consolidation process, etc (see [4], [8], [15]).

Well-posedness of general Dirichlet boundary value problem as well as Cauchy problem for both isotropic and anisotropic degenerate parabolic-hyperbolic equations have been well studied by [12], [16], [19], [21], [22]. However, the Dirichlet boundary condition may not always provide the most natural setting for various physical problems, see [5], [6], [7]. For example, for equations modeling sedimentation-consolidation processes, it's usually appropriate to use kinematic "flux-type" or "wall" boundary conditions rather than the Dirichlet boundary condition (see [6]). In one-dimensional case, specific models (sedimentation-consolidation model) have been considered by Bürger, Evje, Frid and Karlsen [6] [7]. In [7], well-posedness

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has been established under the "flux type" boundary condition using BV approach. It turned out that these boundary conditions are satisfied on the boundary in an a.e. pointwise sense, which is difficult to be extended to the multidimensional case due to lack of regularity on diffusion term $B(u)$. On the other hand, although the BV approach plays an important role in the study of degenerate parabolic equations, it does have many restrictions, see [6]. Based on such considerations, B$\ddot{u}$rger, Frid and Karlsen [6] considered a free boundary problem under the framework of divergence-measure fields which was first considerd by Anzellotti [2] and reformulated by Chen and Frid [10] [11]. It should be noted that the boundary conditions proposed in [6] is too weak to ensure the uniqueness of entropy solutions. It motivated us to impose some stronger boundary condition but still under the framework of divergence-measure fields. Moreover, to our knowledge, there are few results referring to Neumann boundary conditions for strongly degenerate parabolic equations in multidimensional case. We point out that for a kind of special weakly degenerate parabolic-hyperbolic equation with $B'(u) > 0$ for $u \neq 0$ and $B'(0) = 0$, Anderson [3] has proved local well-posedness with mixed-boundary value condition. However, since we can not expect any regularity results for strongly degenerate parabolic-hyperbolic equation, the method in [3] can not be used here.

When $B(u)$ is strictly increasing, the above equation is of parabolic type. On this occasion, existence and uniqueness of weak solutions are well-known, see [18]. In the case where $B' \equiv 0$, problem (1.1) is reduced into a nonlinear hyperbolic conservation laws with a zero-flux boundary condition. This kind of problem has been recently studied by B$\ddot{u}$rger, Frid and Karlsen [5], where they introduced a new formulation of entropy solutions and proved its well-posedness. A main feature in this new formulation lies in the existence of strong traces of any L^∞ entropy solutions, which is based on the result of Vasseur [24]. It should be noted that there is a mistake in the proof of uniqueness in [5] which was solved by Hu and Li [14] by using a different approach.

This paper is mainly devoted to giving a proper formulation of entropy solutions for Neumann boundary value problem (1.1) and proving its uniqueness only on rectangle domain. The main difficulty is to explain the meaning of boundary condition $(\mathbf{f}(u) - \nabla B(u)) \cdot \mathbf{n}(\mathbf{x}, \mathbf{t}) = \mathbf{0}$. Since $u_0(x)$ is only supposed to be bounded in domain Ω, one can not expect the solution $u(\mathbf{x}, t)$ belonging to a more regular space than L^∞ space. Therefore, the meaning of $\mathbf{f}(u) - \nabla B(u)$ on the boundary is not clear. Fortunately, by theory of divergence-measure field introduced by Chen and Frid [10], [11], the vector field $(u, \mathbf{f}(u) - \nabla B(u)) \in DM^2((0, T) \times \Omega)$ has normal trace on the boundary. Thus, the boundary condition $(1.1)_3$ can be understood as that the normal trace of $f(u) - \nabla B(u)$ is zero on the boundary. However, as far as we know, there are no results concerning uniqueness of entropy solutions under such boundary condition. Therefore, we impose some stronger conditions on the boundary which in turn need some stronger regularity for solutions, that is, the existence of strong traces on the boundary. For this purpose, we impose so-called

nonlinearity-diffusivity conditions on the flux function $\mathbf{f}(u)$ and diffusive function $B(u)$. This condition is used to ensure the existence of boundary traces via the result of Kwon [17] as well as the compactness via the result of Lions, Perthame and Tadmor [20]. Our boundary condition is reasonable since it can be derived though an entropy inequality, and thus can be understood as an entropy boundary condition. We also point out that if the solution u is regular, our boundary condition is consistent with classical boundary conditions.

The reason why we consider rectangle domains is as follows: The boundary condition we imposed is still too weak to ensure uniqueness of entropy solutions for general domains. The special structure of rectangle domains allows us to understand the boundary condition in an a.e. pointwise sense in some part of the boundary $\partial\Omega$. Therefore, the boundary terms can be dealt with and uniqueness can be derived.

The existence of entropy solutions is also discussed. For some special flux function $\mathbf{f}$ and diffusion function B, we can show that the entropy solution exists by using the method of vanishing viscosity. More specifically, we assume that there is a critical value u_c such that $B'(\xi) = 0$ for $\xi < u_c$ and $B'(\xi) > 0$ for $\xi > u_c$ and $\mathbf{f}(\xi)$ vanishes on $\xi > u_c$. Here we do not claim that the flux function and the diffusion function satisfying these conditions are appropriate for any real physical models. However, they do satisfy our assumptions for well-posedness of problem (1.1). We also note that such chosen $B(\xi)$ is natural for sedimentation-consolidation processes, see [6], [7].

We organize our paper as follows. In section 2, we state some technical assumptions on the flux $\mathbf{f}(u)$ and introduce the concept of domain with Lipschitz deformable boundary as well as the notations of boundary-layer sequence and normal traces of divergence-measure field. Meanwhile, we present a strong trace results mainly from Kwon [17]. In section 3, we give the definition of entropy solutions to problem (1.1) and prove its uniqueness. In section 4, we discuss the existence of entropy solutions.

2. Preliminaries

We need some assumptions on the flux $\mathbf{f}(u)$ as in [5].

Assumption 2.1. *Assume that $u_0 \in [0, M]$ for fixed $M > 0$ and $\mathbf{f}(u)$ depends smoothly on u for $u \in [0, M]$. Moreover, we require that*

$$\mathbf{f}(0) = 0, \quad \mathbf{f}(M) = 0. \tag{2.1}$$

Assumption 2.2. *Assume that the flux function $\mathbf{f}(u)$ and the diffusion function $B(u)$ satisfy the nonlinearity-diffusivity condition, that is, $\forall(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^N, \tau^2 + |\xi|^2 = 1$,*

$$\mathcal{L}(\{u \in [0, M] \,|\, \tau + \xi \cdot \mathbf{f}'(u) = 0 \quad \text{and} \quad B'(u)|\xi|^2 = 0\}) = 0, \tag{2.2}$$

where $\mathcal{L}$ denotes the one-dimensional Lebesgue measure.

Remark 2.1. *As in [5], Assumption 2.1 is used to ensure L^∞ bounds on the solutions.*

Remark 2.2. *The nonlinearity-diffusivity condition implies that there are no interval of u on which both $\mathbf{f}(u)$ is affine and $B(u)$ is degenerate. This condition is mainly developed from* [13, 20] *and is always used to the theory of compensated compactness* [23]. *Moveover, as is shown in Lemma 2.1, the condition* (2.2) *implies the existence of strong traces of entropy solutions.*

2.1. **Deformable Lipschitz boundary and Strong traces.** Now we introduce the concept of deformable Lipschitz boundary [10] and the notion of strong traces.

Definition 2.1. *We say that $\partial\Omega$ is a deformable Lipschitz boundary provided that the following conditions hold:*
(1) For all $\mathbf{x} \in \partial\Omega$ there exists a number $r > 0$ and a Lipschitz map $h : \mathbb{R}^{N-1} \to \mathbb{R}$ such that, after rotating and relabeling coordinates if necessary,

$$\Omega \cap \mathcal{Q}(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^N : h(y_1, \cdots, y_{N-1}) < y_N\} \cap \mathcal{Q}(\mathbf{x}, r),$$

where $\mathcal{Q}(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^N : |x_i - y_i| \le r,\, i = 1, \cdots, N\}$. We denote by $\tilde{h}$ the map $(y_1, \cdots, y_{N-1}) =: \tilde{y} \mapsto (\tilde{y}, h(\tilde{y}))$.
(2) There exists a map $\Psi : \partial\Omega \times [0, 1] \to \overline{\Omega}$ such that Ψ is a homeomorphism, that is, bi-Lipschitz over its image with $\Psi(\omega, 0) = \omega$ for all $\omega \in \partial\Omega$.
The map Ψ is called a Lipschitz deformation of the boundary $\partial\Omega$. We denote $\Psi_s(\omega) = \Psi(\omega, s)$ and $\partial\Omega_s = \Psi_s(\partial\Omega)$. We also denote by Ω_s the bounded open set whose boundary is $\partial\Omega_s$.

Moreover, the Lipschitz deformation Ψ is said to be *regular* if

$$\lim_{s\to 0+} \nabla\Psi_s \circ \tilde{h} = \nabla\tilde{h} \qquad \text{in } L^1_{loc}(B), \tag{2.3}$$

where B denotes the greatest open set such that $\tilde{h}(B) \subset \partial\Omega$.

The meaning of strong trace is stated in the following sense:

Definition 2.2. *Let $\mathcal{M} \subset \mathbb{R}^{N+1}$ have a regular deformable Lipschitz boundary. We say that a given function $u \in L^\infty(\mathcal{M})$ possesses a strong trace u^τ on $\partial\mathcal{M}$ if $u^\tau \in L^\infty(\partial\mathcal{M})$ have the property that for every regular (with respect to $\partial\mathcal{M}$) Lipschitz deformation Ψ and every compact set $K \subset\subset \partial\mathcal{M}$,*

$$ess \lim_{s\to 0} \int_K |u(\Psi(s, \mathbf{x})) - u^\tau(\mathbf{x})| \mathrm{d}\mathcal{H}^N(\mathbf{x}) = 0, \tag{2.4}$$

where $\mathcal{H}^N$ is the N-dimensional Hausdorff measure.

From now on in this paper, it is always understood that $\Omega \subset \mathbb{R}^N$ is an open bounded rectangle and thus has deformable Lipschitz boundary, see remarks in [10]. Moveover, for each $T > 0$, we denote by $\mathcal{Q}_T$, Σ_T, $\overline{\mathcal{Q}}_T$ the set $\Omega \times (0, T)$, $\partial\Omega \times (0, T)$ and $\overline{\Omega} \times [0, T)$, respectively. The following Lemma gives the existence of strong trace of any entropy solutions to degenerate parabolic equations.

Lemma 2.1. *Let Assumption 2.2 hold. Suppose that $u \in L^\infty(\mathcal{Q}_T)$ and $B(u) \in L^2([0,T]; H^1(\Omega))$ satisfying*

$$\partial_t u + \nabla \cdot \mathbf{f}(u) = \triangle B(u) \quad \text{in}\, \mathcal{D}'(\mathcal{Q}_T)$$

and

$$\partial_t \eta(u) + \nabla \cdot \mathbf{q}(u) - \triangle p(u) \leq 0 \quad \text{in}\, \mathcal{D}'(\mathcal{Q}_T),$$

for any entropy-entropy flux triple $(\eta, \mathbf{q}, p)$ where $\mathbf{q}' = \eta' \mathbf{f}', p' = \eta' B'$. Then u possesses a strong trace u^τ on the boundary.

Remark 2.3. *We notice that under the nonlinearity-diffusivity condition, the proof of this lemma is quite the same with the proof of Theorem 1.1 in* [17]*. Indeed, under condition* (2.2)*, the flux function $\mathbf{f}$ is genuinely nonlinear on the interval where the diffusion function B is degenerate. Thus, combining the technique used in* [17] *and utilizing the results of Vasseur* [24]*, we can easily show that u possesses a strong trace on the boundary.*

2.2. **Boundary-layer sequence ζ_δ and Normal Traces.** In order to deal with the boundary terms, we introduce the concept of boundary-layer sequence, see [21].

Definition 2.3. *We call $\{\zeta_\delta\}$ a boundary-layer sequence if $\zeta_\delta \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ is a sequence of functions such that*

$$\lim_{\delta \to 0+} \zeta_\delta(\mathbf{x}) = 1 \quad \forall \mathbf{x} \in \Omega; \quad 0 \leq \zeta_\delta \leq 1; \quad \zeta_\delta = 0 \quad \text{on} \quad \partial\Omega.$$

Remark 2.4. *For any boundary-layer sequence ζ_δ, $-\nabla\zeta_\delta$ converges to the outward normal $\mathbf{n}$ of the boundary, i.e., for any $\psi \in (H^1(\Omega))^N$,*

$$\lim_{\delta \to 0+} \int_\Omega \psi \cdot \nabla \zeta_\delta = \lim_{\delta \to 0+} \int_\Omega (\nabla \cdot \psi) \zeta_\delta = -\int_\Omega \nabla \cdot \psi = -\int_{\partial\Omega} \psi \cdot \mathbf{n}.$$

Now we introduce the notion of L^2 divergence measure fields given by Chen and Frid in [10, 11], see also [21]. Set

$$\mathcal{DM}^2(\mathcal{Q}_T) := \{\mathbf{F} \in L^2(\mathcal{Q}_T, \mathbb{R}^{N+1}) : \exists C > 0 : \mid \int_{\mathcal{Q}_T} \mathbf{F} \cdot (\partial_t, \nabla) \phi \mathrm{d}x \mathrm{d}t \mid \leq C \|\phi\|_{L^\infty(\mathcal{Q}_T)}\}$$

the space of $L^2(\mathcal{Q}_T)$ vector fields whose divergence is a bounded Radon measure in $\mathcal{Q}_T$. Properties of divergence measure fields have been well studied by Chen and Frid in [10, 11]. In particular, the following lemma which generalize the Gauss-Green formula for divergence measure fields holds, (see also [22]):

Lemma 2.2. *Let $\mathcal{F} \in (L^2(\mathcal{Q}_T))^{N+1}$ be such that $\mathrm{div}\mathcal{F}$ is a bounded Radon measure on $\mathcal{Q}_T$. Then there exists a linear functional $\mathcal{T}_\gamma$ on $W^{\frac{1}{2},2}(\Sigma_T) \cap \mathcal{C}(\Sigma_T)$ which represents the normal trace $\mathcal{F} \cdot \gamma$ on Σ_T in the sense that the following Gauss-Green formula holds: for all $\psi \in \mathcal{C}_0^\infty(\overline{\mathcal{Q}}_T)$,*

$$< \mathcal{T}_\gamma, \psi >= \int_{\mathcal{Q}_T} \psi(\nabla \cdot \mathcal{F}) + \int_{\mathcal{Q}_T} \nabla\psi \cdot \mathcal{F},$$

and $< \mathcal{T}_\gamma, \psi >$ depends only on $\psi|_{\Sigma_T}$.

Let $\{\zeta_\delta\}$ be a boundary-layer sequence, according to discussions in [22] we know that for any $\psi \in \mathcal{C}_0^\infty(\overline{\mathcal{Q}}_T)$

$$< \mathcal{T}_{\mathbf{n}}, \psi >= -\lim_{\varepsilon\to 0}\int_{\mathcal{Q}_T} \tilde{\mathcal{F}}\cdot\nabla\zeta_\delta\psi\mathrm{d}\mathbf{x}\mathrm{d}t, \tag{2.5}$$

where $\tilde{\mathcal{F}} \in (L^2(\mathcal{Q}_T))^N$ is the space part of $\mathcal{F} = (\mathcal{F}_0, \tilde{\mathcal{F}})$ and the linear functional $\mathcal{T}_{\mathbf{n}}$ represents the normal trace $\tilde{\mathcal{F}}\cdot\mathbf{n}$.

3. Uniqueness of Entropy Solutions

In this section, we shall first give a definition of entropy solutions of problem (1.1) which is mainly involved with boundary condition (3.5). Then we show several useful lemmas which are mainly concerned with the techniques in dealing with the boundary terms appeared in the proof of Theorem 3.1. Finally we can prove its uniqueness.

Definition 3.1. *A function $u \in L^\infty(\mathcal{Q}_T)$ is called an entropy solution of the initial-boundary value problem* (1.1) *if the following conditions are satisfied :*
(1)(regularity) The following regularity properties hold:

$$B(u) \in L^2(0,T;H^1(\Omega)) \tag{3.1}$$

$$\forall k\in\mathbb{R}: \quad (|u-k|, \mathrm{sgn}(u-k)(\mathbf{f}(u)-\mathbf{f}(k)) - \nabla|B(u)-B(k)|) \in \mathcal{DM}^2(\mathcal{Q}_T). \tag{3.2}$$

(2) (interior entropy condition) $\forall k\in\mathbb{R}$, $\forall\phi\in\mathcal{C}_0^\infty(\mathcal{Q}_T)$ with $\phi\geq 0$,

$$\int_{\mathcal{Q}_T} |u-k|\partial_t\phi + \big\{\mathrm{sgn}(u-k)(\mathbf{f}(u)-\mathbf{f}(k)) - \nabla|B(u)-B(k)|\big\}\cdot\nabla\phi\mathrm{d}\mathbf{x}\mathrm{d}t \geq 0; \tag{3.3}$$

(3)(initial condition) The initial condition is satisfied as a limit in the following L^1 sense:

$$ess\lim_{t\to 0+}\int_\Omega |u(\mathbf{x},t)-u_0(\mathbf{x})|\mathrm{d}\mathbf{x} = 0; \tag{3.4}$$

(4) (boundary condition) For any boundary-layer sequence $\{\xi_\delta\}$ and nonnegative function $\varphi\in\mathcal{C}_0^\infty(\overline{\mathcal{Q}}_T)$, $\forall k\in\mathbb{R}$,

$$\lim_{\delta\to 0}\int_{\mathcal{Q}_T}\mathrm{sgn}(u-k)(\nabla B(u)-\mathbf{f}(u))\cdot\nabla_{\mathbf{x}}\xi_\delta\varphi\mathrm{d}\mathbf{x}\mathrm{d}t \geq 0. \tag{3.5}$$

Remark 3.1. *By the theory of divergence-measure field, the entropy-entropy flux pair*

$$(|u-k|, \mathrm{sgn}(u-k)\,(\mathbf{f}(u)-\mathbf{f}(k)-\nabla(B(u)-B(k))))$$

has normal traces for any $k\in\mathbb{R}$, i.e., the limit

$$\lim_{\delta\to 0}\int_{\mathcal{Q}_T}\mathrm{sgn}(u-k)(\mathbf{f}(u)-\mathbf{f}(k)-\nabla(B(u)-B(k)))\cdot\nabla_{\mathbf{x}}\xi_\delta\varphi\mathrm{d}\mathbf{x}\mathrm{d}t \tag{3.6}$$

exists. On the other hand, by Lemma 2.1, the limit

$$\lim_{\delta\to 0}\int_{\mathcal{Q}_T}\operatorname{sgn}(u-k)\mathbf{f}(k)\cdot\nabla_{\mathbf{x}}\xi_\delta\varphi\mathrm{d}\mathbf{x}\mathrm{d}t \tag{3.7}$$

exists. Therefore, from (3.6) *and* (3.7)*, the limit in* (3.5) *exists.*

Remark 3.2. *Taking* $k=\pm\|u\|_{L^\infty}$ *in* (3.5)*, we get*

$$\lim_{\delta\to 0}\int_{\mathcal{Q}_T}(\mathbf{f}(u)-\nabla B(u))\cdot\nabla_{\mathbf{x}}\xi_\delta\varphi\mathrm{d}\mathbf{x}\mathrm{d}t=0. \tag{3.8}$$

It seems more natural to impose the condition (3.8) *as the boundary condition. But it turns out to be too weak to ensure the uniqueness of entropy solutions (see* [6]*). On the other hand, if the function* $\mathbf{f}(u)-\nabla B(u)$ *has strong trace on the boundary, the equality* (3.8) *reduces into the classical boundary condition*

$$(\mathbf{f}(u)-\nabla B(u))\cdot\mathbf{n}=0 \qquad \text{on} \quad \partial\Omega. \tag{3.9}$$

Remark 3.3. *If the problem* (1.1) *is fully degenerate, i.e,* $B'(u)\equiv 0$*, the definition above is exactly the same as the definition of entropy solutions for zero-flux problem of conservation laws in* [5].

Remark 3.4. *It is easy to show that* (3.3)*,* (3.4) *and* (3.5) *are equivalent to the following inequality:* $\forall k\in R, \forall\varphi\in\mathcal{C}_0^\infty(\overline{\mathcal{Q}}_T)$ *with* $\varphi\geq 0$*,*

$$\begin{aligned}&\int_{\mathcal{Q}_T}|u-k|\partial_t\varphi+\big\{\operatorname{sgn}(u-k)(\mathbf{f}(u)-\mathbf{f}(k))-\nabla|B(u)-B(k)|\big\}\nabla\varphi\mathrm{d}\mathbf{x}\mathrm{d}t+\\&\int_{\Omega}|u_0(\mathbf{x})-k|\varphi_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^T\int_{\partial\Omega}\operatorname{sgn}(u^\tau-k)\mathbf{f}(k)\cdot\mathbf{n}\varphi\mathrm{d}\mathcal{H}^{N-1}\mathrm{d}t\geq 0,\end{aligned} \tag{3.10}$$

where $\varphi_0(\mathbf{x})=\varphi(0,\mathbf{x})$ *and* u^τ *is the strong trace of* u *on the boundary. In fact,* (3.10) *implies* (3.3) *and* (3.4) *obviously. Take* $\varphi=(1-\zeta_\delta)\psi$ *with* $\psi\in\mathcal{C}_0^\infty(\overline{\mathcal{Q}}_T)$ *in* (3.10) *and let* $\delta\to 0$*, the inequality* (3.5) *follows immediately. On the other hand, taking* $\phi=\zeta_\delta\varphi$ *in* (3.3) *where* $\varphi\in\mathcal{C}_0^\infty(\overline{\mathcal{Q}}_T)$ *and using boundary condition* (3.5)*, we get* (3.10) *immediately.*

The following two lemmas are useful in the proof of uniqueness of entropy solutions.

Lemma 3.1. *Let* u *be an entropy solution defined in Definition 3.1 and* $D\subset\mathbb{R}^N$ *be an open set such that* $D\cap\partial\Omega=\{\mathbf{x}=(\tilde{x},x_N)|x_N=0\}$ *and* $D\cap\Omega=\{\mathbf{x}=(\tilde{x},x_N)|x_N<0\}$*. Then for any boundary-layer sequence* $\{\zeta_\delta\}$ *and nonnegative function* $\psi\in\mathcal{C}_0^\infty(D\times(0,T))$*,* $\forall A(\tilde{x},t)\in L^\infty(D\cap\partial\Omega\times(0,T))$*, the following inequality holds:*

$$\liminf_{\delta\to 0}\int_{\mathcal{Q}_T}\operatorname{sgn}(u-A(\tilde{x},t))(\nabla B(u)-\mathbf{f}(u))\cdot\nabla\zeta_\delta\psi\mathrm{d}\mathbf{x}\mathrm{d}t\geq 0. \tag{3.11}$$

Proof. First we claim that for any simple function $S(\tilde{x},t) = \sum_{i=1}^{m} \mathbf{1}_{E_i\times(0,T)} k_i$ where $k_i \in \mathbb{R}$, $\cup_{i=1}^m E_i = D \cap \partial\Omega$, and

$$\mathbf{1}_{E_i\times(0,T)} = \begin{cases} 1, & (x,t) \in E_i \times (0,T) \\ 0, & otherwise \end{cases}$$

the following inequality holds:

$$\int_{Q_T} |u - S(\tilde{x},t)|\partial_t\varphi + \big\{\mathrm{sgn}(u - S(\tilde{x},t))(\mathbf{f}(u) - \mathbf{f}(S(\tilde{x},t)) - \nabla B(u))\big\}\nabla\varphi \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_{\partial\Omega} \mathrm{sgn}(u^\tau - S(\tilde{x},t))\mathbf{f}(S(\tilde{x},t)) \cdot \mathbf{n}\varphi \mathrm{d}\mathcal{H}^{N-1}\mathrm{d}t \geq 0, \quad (3.12)$$

where $\varphi \in \mathcal{C}_0^\infty(D \times (0,T))$. In fact, we only have to show that for each i,

$$\int_0^T \int_{x_N} \int_{E_i} |u - k_i|\partial_t\varphi + \big\{\mathrm{sgn}(u - k_i)(\mathbf{f}(u) - \mathbf{f}(k_i) - \nabla B(u))\big\}\nabla\varphi \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_{E_i} \mathrm{sgn}(u^\tau - k_i)\mathbf{f}(k_i) \cdot \mathbf{n}\varphi \mathrm{d}\mathcal{H}^{N-1}\mathrm{d}t \geq 0. \quad (3.13)$$

This follows easily by taking $k = k_i$ and $\varphi = \mathbf{1}_{E_i}\psi$ in (3.10). Now, suppose $A(\tilde{x},t) \in L^\infty(D \cap \partial\Omega \times (0,T))$, then there exists a sequence of simple functions $\{A_n(\tilde{x},t)\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} A_n(\tilde{x},t) = A(\tilde{x},t) \quad \text{a.e.} \quad \text{for} \quad (\tilde{x},t) \in D \cap \partial\Omega \times (0,T).$$

For each A_n, we know that (3.12) holds. Let $n \to \infty$, we immediately get

$$\int_{Q_T} |u - A(\tilde{x},t)|\partial_t\varphi + \big\{\mathrm{sgn}(u - A(\tilde{x},t))(\mathbf{f}(u) - \mathbf{f}(A(\tilde{x},t)) - \nabla B(u))\big\}\nabla\varphi \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_{\partial\Omega} \mathrm{sgn}(u^\tau - A(\tilde{x},t))\mathbf{f}(A(\tilde{x},t)) \cdot \mathbf{n}\varphi \mathrm{d}\mathcal{H}^{N-1}\mathrm{d}t \geq 0. \quad (3.14)$$

Taking $\varphi = (1-\zeta_\delta)\psi$ in (3.14) where $\psi \in \mathcal{C}_0^\infty(D \times (0,T))$ and letting $\delta \to 0$, we get (3.11) immediately. □

The next lemma comes from [1] and is modified in our case as follows:

Lemma 3.2. *Let D be a bounded subset in $\mathbb{R}^N$ and $\omega_1, \omega_2, J_1, J_2 \in L^1(D)$. Let $\rho_i(z) = i\rho(iz)$ where $\rho \in \mathcal{C}_0^\infty([-1,1],\mathbb{R})$ is a nonnegative even function such that $\int_{\mathbb{R}} \rho(r)\mathrm{d}r = 1$. Then*

$$\limsup_{i\to\infty} \iint_{D\times D} \mathrm{sgn}(\omega_1(x) - \omega_2(y))(J_1(x) - J_2(y))\rho_i(x-y)\mathrm{d}x\mathrm{d}y \leq \int_D \mathrm{sgn}(\omega_1 - \omega_2)(J_1 - J_2)\mathrm{d}x + \int_{\{\omega_1=\omega_2\}} |J_1 - J_2|\mathrm{d}x.$$

Moreover, if $J_1 = J_2$ a.e. on $\omega_1 = \omega_2$, then there exists

$$\lim_{i\to\infty}\iint_{D\times D}\operatorname{sgn}(\omega_1(x)-\omega_2(y))(J_1(x)-J_2(y)\rho_i(x-y)\mathrm{d}x\mathrm{d}y = \int_D \operatorname{sgn}(\omega_1-\omega_2)(J_1-J_2)\mathrm{d}x. \tag{3.15}$$

Now we can show the uniqueness of the entropy solutions to problem (1.1).

Theorem 3.1. *Suppose that $u(\mathbf{x},t)$ and $v(\mathbf{y},s)$ are entropy solutions of initial boundary value problem* (1.1) *with initial data u_0 and v_0, respectively. Then for a.e. $t>0$,*

$$\int_\Omega |u(\mathbf{x},t)-v(\mathbf{x},t)|\mathrm{d}\mathbf{x} \le \int_\Omega |u_0(\mathbf{x})-v_0(\mathbf{x})|\mathrm{d}\mathbf{x}. \tag{3.16}$$

In particular there exists at most one entropy solution to the initial boundary value problem (1.1).

Proof. Let $\xi(\mathbf{x},t,\mathbf{y},s)$ be nonnegative smooth function which satisfies,

$$\xi(\cdot,\cdot,\mathbf{y},s)\in\mathcal{C}_0^\infty(\mathcal{Q}_T) \quad \text{for} \quad \text{fixed} \quad (\mathbf{y},s)\in\mathcal{Q}_T,$$

$$\xi(\mathbf{x},t,\cdot,\cdot)\in\mathcal{C}_0^\infty(\mathcal{Q}_T) \quad \text{for} \quad \text{fixed} \quad (\mathbf{x},t)\in\mathcal{Q}_T.$$

By standard "doubling of variables" (see [9]), we get for test function $\xi(\mathbf{x},t,\mathbf{y},s)$

$$\begin{aligned}\iint_{\mathcal{Q}_T\times\mathcal{Q}_T} &\{|u(\mathbf{x},t)-v(\mathbf{y},s)|(\xi_t+\xi_s)\\ &+\operatorname{sgn}(u(\mathbf{x},t)-v(\mathbf{y},s))(f(u(\mathbf{x},t))-f(v(\mathbf{y},s)))\cdot(\nabla_\mathbf{x}\xi+\nabla_\mathbf{y}\xi)\\ &-(\nabla_\mathbf{x}|B(u)(\mathbf{x},t)-B(v)(\mathbf{y},s)|+\nabla_\mathbf{y}|B(u)(\mathbf{x},t)-B(v)(\mathbf{y},s)|)\cdot(\nabla_\mathbf{x}\xi+\nabla_\mathbf{y}\xi)\}\mathrm{d}z\ge 0,\end{aligned}$$

where $\mathrm{d}z = \mathrm{d}\mathbf{x}\mathrm{d}t\mathrm{d}\mathbf{y}\mathrm{d}s$.

For $i=1,2$, let $\psi_i\in\mathcal{C}_0^\infty(\mathbb{R}^N), 0\le\psi_i\le 1$ with $\operatorname{supp}\psi_1\subset\subset D$, where $D\subset\mathbb{R}^N$ is an open set such that

$$D\cap\partial\Omega = \{\mathbf{x}=(\tilde{x},x_N)|x_N=0\}$$

and

$$D\cap\Omega = \{\mathbf{x}=(\tilde{x},x_N)|x_N<0\}.$$

We also take $\psi_2\equiv 1$ on the support of ψ_1 such that $\psi_1(\mathbf{x})\psi_2(\mathbf{x})=\psi_1(\mathbf{x})$. We denote $\psi_1(\mathbf{x})\psi_2(\mathbf{y})$ by $\psi(\mathbf{x},\mathbf{y})$. Choose $\theta(t)\in\mathcal{C}_0^\infty(0,T),\theta\ge 0$. Let $\xi(\mathbf{x},t,\mathbf{y},s) = \zeta_\delta(\mathbf{x})\zeta_\eta(\mathbf{y})\rho_l(t-s)\rho_m(\tilde{x}-\tilde{y})\rho_n(x_N-y_N)\theta(t)\psi(\mathbf{x},\mathbf{y})$, where ζ_δ,ζ_η are boundary-layer sequences and ρ_l,ρ_m,ρ_n are sequences of mollifiers and $\mathbf{x}=(\tilde{x},x_N),\mathbf{y}=(\tilde{y},y_N)$ are

local coordinates induced by ψ. Then we get

$$\begin{aligned}
&\iint\limits_{\mathcal{Q}_T\times\mathcal{Q}_T}\{|u-v|\zeta_\delta\zeta_\eta\rho_{l,m,n}\psi\theta'+\mathrm{sgn}(u-v)(\mathbf{f}(u)-\mathbf{f}(v))\cdot\nabla_{\mathbf{x}+\mathbf{y}}\psi\zeta_\delta\zeta_\eta\rho_{l,m,n}\theta\\
&-(\nabla_{\mathbf{x}}|B(u)-B(v)|+\nabla_{\mathbf{y}}|B(u)-B(v)|)\cdot\nabla_{\mathbf{x}+\mathbf{y}}\psi\zeta_\delta\zeta_\eta\rho_{l,m,n}\theta\}\mathrm{d}z\\
&\geq-\iint\limits_{\mathcal{Q}_T\times\mathcal{Q}_T}\mathrm{sgn}(u-v)(\mathbf{f}(u)-\mathbf{f}(v)-\nabla_{\mathbf{x}}B(u))\cdot\nabla_{\mathbf{x}}\zeta_\delta\zeta_\eta\rho_{l,m,n}\theta\psi\mathrm{d}z\\
&-\iint\limits_{\mathcal{Q}_T\times\mathcal{Q}_T}\mathrm{sgn}(u-v)(\mathbf{f}(u)-\mathbf{f}(v)+\nabla_{\mathbf{y}}B(u))\cdot\nabla_{\mathbf{y}}\zeta_\eta\zeta_\delta\rho_{l,m,n}\theta\psi\mathrm{d}z\\
&+\iint\limits_{\mathcal{Q}_T\times\mathcal{Q}_T}\nabla_{\mathbf{x}}|B(u)-B(v)|\cdot\nabla_{\mathbf{y}}\zeta_\eta\zeta_\delta\rho_{l,m,n}\theta\psi\mathrm{d}z\\
&+\iint\limits_{\mathcal{Q}_T\times\mathcal{Q}_T}\nabla_{\mathbf{y}}|B(u)-B(v)|\cdot\nabla_{\mathbf{x}}\zeta_\delta\zeta_\eta\rho_{l,m,n}\theta\psi\mathrm{d}z\\
&=:J_1+J_2+J_3+J_4,
\end{aligned}$$

where we denote $\rho_l\rho_m\rho_n$ by $\rho_{l,m,n}$ for simplicity. Let

$$J_1=\tilde{J}_1+\iint\limits_{\mathcal{Q}_T\times\mathcal{Q}_T}\mathrm{sgn}(u-v)\mathbf{f}(v)\cdot\nabla_{\mathbf{x}}\zeta_\delta\zeta_\eta\theta\psi\rho_{l,m,n}\mathrm{d}z,$$

where

$$\tilde{J}_1=\iint\limits_{\mathcal{Q}_T\times\mathcal{Q}_T}\mathrm{sgn}(u-v)(\nabla_{\mathbf{x}}B(u)-\mathbf{f}(u))\cdot\nabla_{\mathbf{x}}\zeta_\delta\zeta_\eta\theta\psi\rho_{l,m,n}\mathrm{d}z.$$

For a.e. $(\mathbf{y},s)\in\mathcal{Q}_T$, let

$$F_\delta(\mathbf{y},s)=\int_{\mathcal{Q}_T}\mathrm{sgn}(u(\mathbf{x},t)-v(\mathbf{y},s))(\nabla_{\mathbf{x}}B(u)-\mathbf{f}(u))\cdot\nabla_{\mathbf{x}}\zeta_\delta\zeta_\eta\theta\psi\rho_{l,m,n}\mathrm{d}\mathbf{x}\mathrm{d}t.$$

By entropy boundary condition (3.5), we have for a.e. $(\mathbf{y},s)\in\mathcal{Q}_T$,

$$\lim_{\delta\to0}F_\delta(\mathbf{y},s)\geq0. \tag{3.17}$$

So, by Fatou's lemma, we get

$$\liminf_{\delta\to0}\tilde{J}_1\geq\int_{\mathcal{Q}_T}\liminf_{\delta\to0}F_\delta(\mathbf{y},s)\mathrm{d}\mathbf{y}\mathrm{d}s\geq0. \tag{3.18}$$

Therefore, we conclude that

$$\liminf_{\eta,\delta\to0}J_1\geq-\iint\limits_{\Sigma_x\times\mathcal{Q}_T^y}sgn(u^\tau-v)\mathbf{f}(v)\cdot\mathbf{n}_x\theta\psi\rho_{l,m,n}\mathrm{d}\mathcal{H}^{N-1}\mathrm{d}t\mathrm{d}\mathbf{y}\mathrm{d}s,$$

where $\Sigma_{\mathbf{x}}$ is the boundary domain with respect to $\mathbf{x}$, $\mathbf{n_x}$ is the outward normal vector to $\Sigma_{\mathbf{x}}$ and u^τ is strong trace of u on the boundary. Similarly, we obtain

$$\liminf_{\delta,\eta\to 0} J_2 \geq \iint\limits_{\mathcal{Q}_T^x\times\Sigma_y} \operatorname{sgn}(u-v^\tau)\mathbf{f}(u)\cdot\mathbf{n}_y\theta\psi\rho_{l,m,n}\mathrm{d}\mathbf{x}\mathrm{d}t\mathrm{d}\mathcal{H}^{N-1}\mathrm{d}s,$$

$$\lim_{\delta,\eta\to 0} J_3 = -\iint\limits_{\mathcal{Q}_T^x\times\Sigma_y} \operatorname{sgn}(u-v^\tau)\nabla_{\mathbf{x}}B(u)\cdot\mathbf{n}_y\theta\psi\rho_{l,m,n}\mathrm{d}\mathbf{x}\mathrm{d}t\mathrm{d}\mathcal{H}^{N-1}\mathrm{d}s,$$

$$\lim_{\delta,\eta\to 0} J_4 = \iint\limits_{\mathcal{Q}_T^y\times\Sigma_{\mathbf{x}}} \operatorname{sgn}(u^\tau-v)\nabla_{\mathbf{y}}B(v)\cdot\mathbf{n}_x\theta\psi\rho_{l,m,n}\mathrm{d}\mathbf{y}\mathrm{d}s\mathrm{d}\mathcal{H}^{N-1}\mathrm{d}t.$$

Thus we get

$$\begin{aligned}&\iint\limits_{\mathcal{Q}_T\times\mathcal{Q}_T}\{|u-v|\rho_l\rho_m\rho_n\psi\theta' + \operatorname{sgn}(u-v)(\mathbf{f}(u)-\mathbf{f}(v))\cdot\nabla_{\mathbf{x}+\mathbf{y}}\psi\rho_{l,m,n}\theta\\ &-(\nabla_{\mathbf{x}}|B(u)-B(v)|+\nabla_{\mathbf{y}}|B(u)-B(v)|)\cdot\nabla_{\mathbf{x}+\mathbf{y}}\psi\rho_{l,m,n}\theta\}\mathrm{d}z\\ &\geq J_5+J_6,\end{aligned}$$

where

$$J_5 = -\iint\limits_{\mathcal{Q}_T^x\times\Sigma_y} \operatorname{sgn}(u-v^\tau)(\nabla_{\mathbf{x}}B(u)-\mathbf{f}(u))\cdot\mathbf{n}_y\theta\psi\rho_{l,m,n}\mathrm{d}\mathbf{x}\mathrm{d}t\mathrm{d}\mathcal{H}^{N-1}\mathrm{d}s,$$

and

$$J_6 = \iint\limits_{\mathcal{Q}_T^y\times\Sigma_x} \operatorname{sgn}(u^\tau-v)(\nabla_{\mathbf{y}}B(v)-\mathbf{f}(v))\cdot\mathbf{n}_x\theta\psi\rho_{l,m,n}\mathrm{d}\mathbf{y}\mathrm{d}s\mathrm{d}\mathcal{H}^{N-1}\mathrm{d}t.$$

Next let $l,m\to\infty$, using lemma 3.2, we obtain

$$\begin{aligned}\liminf_{l,m\to\infty} J_5 &= -\limsup_{l,m\to\infty}\iint\limits_{\mathcal{Q}_T^x\times\Sigma_y} \operatorname{sgn}(u-v^\tau)(\nabla_{\mathbf{x}}B(u)-\mathbf{f}(u))\cdot\mathbf{n}_y\theta\psi\rho_{l,m,n}\mathrm{d}\mathbf{x}\mathrm{d}t\mathrm{d}\mathcal{H}^{N-1}\mathrm{d}s,\\ &\geq \int_{\mathcal{Q}_T}\operatorname{sgn}(u-v^\tau)(\nabla_{\mathbf{x}}B(u)-\mathbf{f}(u))\cdot(\mathbf{0},-1)\theta\psi_1\rho_n(x_N)\mathrm{d}\tilde{x}\mathrm{d}x_N\mathrm{d}t\\ &-\int_E|(\nabla_x B(u)-f(u))\cdot(\mathbf{0},-1)\theta\psi_1\rho_n(x_N)|\mathrm{d}\tilde{x}\mathrm{d}x_N\mathrm{d}t\\ &:= J_7+J_8,\end{aligned}$$

where $E := \{(\tilde{x},x_N)|u(\tilde{x},x_N) = v^\tau(\tilde{x})\}$. Let $\omega_n = 2\int_{x_N}^0\rho_n(\xi)\mathrm{d}\xi$, then we know $\nabla\omega_n = 2(\mathbf{0},-1)\rho_n(x_N)$. Therefore, using lemma 3.1, we get

$$\liminf_{n\to\infty} J_7 = \frac{1}{2}\liminf_{n\to\infty}\int_{\mathcal{Q}_T}\operatorname{sgn}(u-v^\tau)(\nabla B(u)-\mathbf{f}(u))\cdot\nabla\omega_n\theta\psi_1\mathrm{d}\tilde{x}\mathrm{d}x_N \geq 0.$$

Next, we shall use the special property of rectangle domain to show that

$$\liminf_{n\to\infty} J_8 = 0.$$

We first note that

$$\nabla_x B(u)\cdot(\mathbf{0},-1) = -\partial_{x_N} B(u) = 0 \qquad \text{on} \quad E.$$

On the other hand, we can easily show that

$$\liminf_{n\to\infty} \int_E |f_N|\theta\rho_n(x_N)\psi_1 \mathrm{d}\mathbf{x} = 0. \tag{3.19}$$

Indeed, from remark 3.2, $\forall \psi \in \mathcal{C}_0^\infty((E\backslash\partial\Omega)\cap D)$, we have

$$\liminf_{n\to\infty} \int_E f_N \rho_n(x_N)\psi = 0,$$

which implies $f_N = 0$ a.e. on $\partial E\cap\partial\Omega\cap D$. Thus, (3.19) holds. Based on the above calculations, we get

$$\liminf_{n\to\infty}\liminf_{l,m\to\infty} J_5 \geq 0. \tag{3.20}$$

Similarly, we have $\liminf\limits_{n\to\infty}\liminf\limits_{l,m\to\infty} J_6 \geq 0$. Collecting all the limits above, we obtain the following inequality:

$$\begin{aligned}\int_{\mathcal{Q}_T} &|u-v|\psi_1(\mathbf{x})\theta' + \operatorname{sgn}(u-v)(\mathbf{f}(u)-\mathbf{f}(v))\cdot\nabla_{\mathbf{x}}\psi_1\theta \\ &\qquad -(\nabla_{\mathbf{x}}|B(u)-B(v)|)\cdot\nabla_{\mathbf{x}}\psi_1\theta \mathrm{d}\mathbf{x}\mathrm{d}t \geq 0. \end{aligned}\tag{3.21}$$

Now, we shall do the same partition of $\overline{\Omega}$ as in [16]. Let $\Omega = \prod_{i=1}^N (a_i^-, a_i^+)$ be an open bounded rectangle with $2N$ faces

$$(\partial\Omega)_{i^*} = \{(x_1,\cdots,x_{i-1},a_i^*,x_{i+1},\cdots,x_d); a_j^- < x_j < a_j^+ \quad \text{for} \quad j=1,2,\cdots,d, j\neq i\}$$

and the outward normal $\mathbf{n}_{i^*}$ to Ω along $(\partial\Omega)_{i^*}$ for $i\in\{1,2,\cdots,d\}$, where the super-index $*$ denotes the symbol $+$ or $-$. We set $\Sigma_{i^*} = (0,T)\times(\partial\Omega)_{i^*}$. Set $J = \{1^+,\cdots,N^+,1^-,\cdots,N^-\}$ and $J_0 = \{0\}\cup J$. For $\nu>0$ and $i^*\in J$ we set $U_{i^*}^\nu$, $(\partial\Omega)_{i^*}^\nu$, $\Omega_{i^*}^\nu$, $\tilde{\Omega}_{i^*}^\nu$ and $\triangle_{i^*}^\nu$ as follows: $U_{i^*}^\nu$ is the open subset of all $x\in\Omega$ such that $\operatorname{dist}(x,(\partial\Omega)_{i^*}) \leq \nu$ and $\operatorname{dist}(x,(\partial\Omega)_{i^*}) < \operatorname{dist}(x,(\partial\Omega)_{j^*})$ for all $j\in\{1,2,\cdots,N\}$ with $j\neq i$. $(\partial\Omega)_{i^*}^\nu$ is the subset of all $x\in(\partial\Omega)_{i^*}$ such that $\mathbf{x}-s\mathbf{n}_{i^*}\in U_{i^*}^\nu$ for all $s\in(0,\nu)$. $\Omega_{i^*}^\nu = \{\mathbf{x}-s\mathbf{n}_{i^*}; \mathbf{x}\in(\partial\Omega)_{i^*}^\nu, s\in(0,\nu)\}$, the largest cylinder generated by $\mathbf{n}_{i^*}$ included in $U_{i^*}^\nu$. $\tilde{\Omega}_{i^*}^\nu = \{\mathbf{x}-s\mathbf{n}_{i^*}; \mathbf{x}\in(\partial\Omega)_{i^*}^\nu, s\in(-\nu,\nu)\}$. $\triangle_{i^*}^\nu = U_{i^*}^\nu\backslash\Omega_{i^*}^\nu$. Thus, we have $\operatorname{meas}(\cup_{i^*\in J}\triangle_{i^*}^\nu) \leq \text{Const.}\nu^2$. Moreover, we set $i^*=0$ if $i=0$, $\Omega_0^\nu = \Omega\backslash\cup_{i^*\in J}U_{i^*}^\nu$ and $\Omega^\nu = \cup_{i^*\in J_0}\Omega_{i^*}^\nu$. Since the family $\{U_0^{\frac{\nu}{2}}, \tilde{\Omega}_{i^+}^\nu, \tilde{\Omega}_{i^-}^\nu\}_{i=1}^N$ is an open cover of $\overline{\Omega^{2\nu}}$, we can choose $\{\phi_i\}_{\{0\leq i\leq 2N+1\}}$ a partition of the unity subordinated to the open cover. It's easy to see that for any $j\in\{1,2,\cdots,2N+1\}$, $\operatorname{supp}\phi_j\cap\partial\Omega =$

$\{\mathbf{x} = (\tilde{x}, x_N) | x_N = 0\}$ after change of coordinates. Using (3.21) with ϕ_i in place of ψ_1 and summing on i we get

$$\int_0^T \int_{\overline{\Omega^{2\nu}}} |u(\mathbf{x}, t) - v(\mathbf{x}, t)| \theta' \mathrm{d}\mathbf{x}\mathrm{d}t \geq 0, \tag{3.22}$$

for any $\theta \in \mathcal{C}_0^\infty(0, T)$. Let ν goes to zero, we get

$$\int_{\mathcal{Q}_T} |u(\mathbf{x}, t) - v(\mathbf{x}, t)| \theta' \mathrm{d}\mathbf{x}\mathrm{d}t \geq 0, \tag{3.23}$$

Thus inequality (3.16) follows from (3.23) in a standard way. □

4. Existence of Entropy Solutions

In this section, we prove the existence of entropy solutions for specially chosen $\mathbf{f}$ and B. Specifically, we assume that

$$B'(\xi) = 0, \quad \text{for} \quad 0 \leq \xi \leq u_c; \qquad B'(\xi) > 0, \quad \text{for} \quad u_c < \xi \leq M. \tag{4.1}$$

and

$$\mathbf{f}(\xi) = 0, \quad \text{for} \quad \xi \geq u_c. \tag{4.2}$$

Remark 4.1. *Note that conditions* (4.1) *and* (4.2) *are compatible with the nonlinearity-diffusivity condition* (2.2).

We shall use vanishing viscosity method to prove the existence result. For $\varepsilon > 0$, we consider the following regularized parabolic problem:

$$\begin{cases} \partial_t u^\varepsilon + \nabla \cdot \mathbf{f}(u^\varepsilon) = \triangle B(u^\varepsilon) + \varepsilon \triangle u^\varepsilon, & (\mathbf{x}, t) \in \mathcal{Q}_T, \\ u^\varepsilon(\mathbf{x}, 0) = u_0^\varepsilon(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \\ (\mathbf{f}(u^\varepsilon) - \nabla B(u^\varepsilon) - \varepsilon \nabla u^\varepsilon) \cdot \mathbf{n} = 0, & (\mathbf{x}, t) \in \Sigma_T, \end{cases} \tag{4.3}$$

where u_0^ε is a sequence of smooth functions converging to u_0 in $L^1(\Omega)$ assuming values in $[\varepsilon, M - \varepsilon]$ for $\varepsilon > 0$ sufficiently small. The existence and uniqueness of classical solutions to (4.3) follow from standard arguments, see [[18], Chapter V]. Before we prove the existence of entropy solutions, we need a uniform a priori bound for u^ε.

Lemma 4.1. *Suppose that Assumption 2.1 holds, then the approximate solutions satisfy* $0 \leq u^\varepsilon \leq M$. *Furthermore, there exists a subsequence* $\{\varepsilon_n\}$ *and a function* $u(\mathbf{x}, t) \in L^\infty(\mathcal{Q}_T)$ *such that* $u^{\varepsilon_n} \to u \quad \text{in} \quad L^1_{\mathrm{loc}}(\mathcal{Q}_T)$ *as* $\varepsilon_n \to 0$ *and* $0 \leq u \leq M$.

Proof. Multiplying $(4.3)_1$ by $(u^\varepsilon - M)^+$ and integrating over $\mathcal{Q}_T$, using the boundary condition $(4.3)_3$, we get

$$\int_\Omega |(u^\varepsilon - M)^+|^2 \mathrm{d}\mathbf{x} + \int_{\mathcal{Q}_T} \nabla(B(u) - B(M)) \cdot \nabla(u - M)^+ \mathrm{d}\mathbf{x}\mathrm{d}t$$

$$+\varepsilon \int_{\mathcal{Q}_T} |\nabla(u^\varepsilon - M)^+|^2 \mathrm{d}\mathbf{x}\mathrm{d}t = \int_{\mathcal{Q}_T} \mathbf{f}(u) \cdot \nabla(u^\varepsilon - M)^+ \mathrm{d}\mathbf{x}\mathrm{d}t.$$

By Assumption 2.1 and using Young inequalities, we easily get

$$\int_{\Omega} |(u^\varepsilon - M)^+|^2 \mathrm{d}\mathbf{x} \leq C \int_{\mathcal{Q}_T} |(u^\varepsilon - M)^+|^2 \mathrm{d}\mathbf{x}\mathrm{d}t.$$

By Gronwall's inequality and noting that $\|u_0^\varepsilon\|_{L^\infty} \leq M$, we get $u^\varepsilon \leq M$. By the same approach, using $(u^\varepsilon)^-$ as test function, we get $u^\varepsilon \geq 0$. Therefore, by the compactness result of Lions, Perthame and Tadmor [20], we may extract a subsequence of solutions of (4.3) which converges in $L^1_{\mathrm{loc}}(\mathcal{Q}_T)$ to a function $u(\mathbf{x}, t)$ and $0 \leq u(\mathbf{x}, t) \leq M$. □

Lemma 4.2. *The limit function u of solutions u^ε of the regularized problem* (4.3) *satisfies* (3.1) *and* (3.2) *in Definition 3.1.*

Proof. Multiplying $(4.3)_1$ by $B(u^\varepsilon)$ and integrating over $\mathcal{Q}_T$, we get

$$\begin{aligned}
&\int_{\mathcal{Q}_T} \partial_t u^\varepsilon B(u^\varepsilon) \mathrm{d}\mathbf{x}\mathrm{d}t + \int_{\mathcal{Q}_T} (\nabla \cdot \mathbf{f}(u^\varepsilon)) B(u^\varepsilon) \mathrm{d}\mathbf{x}\mathrm{d}t \\
=& \int_{\mathcal{Q}_T} \triangle B(u^\varepsilon) \cdot B(u^\varepsilon) \mathrm{d}\mathbf{x}\mathrm{d}t + \int_{\mathcal{Q}_T} \varepsilon \triangle u^\varepsilon B(u^\varepsilon) \mathrm{d}\mathbf{x}\mathrm{d}t.
\end{aligned} \tag{4.4}$$

Integrating by part and using $(4.3)_3$, we get

$$\begin{aligned}
&\int_{\mathcal{Q}_T} \partial_t u^\varepsilon B(u^\varepsilon) \mathrm{d}\mathbf{x}\mathrm{d}t - \int_{\mathcal{Q}_T} \mathbf{f}(u^\varepsilon) \cdot \nabla B(u^\varepsilon) \mathrm{d}\mathbf{x}\mathrm{d}t \\
=& \int_{\mathcal{Q}_T} |\nabla B(u^\varepsilon)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t + \int_{\mathcal{Q}_T} \varepsilon \nabla u^\varepsilon \cdot \nabla B(u^\varepsilon) \mathrm{d}\mathbf{x}\mathrm{d}t.
\end{aligned} \tag{4.5}$$

Let A be primitive function of B, i.e., $A' = B$. Then by Lemma 4.1, we get

$$\begin{aligned}
&\frac{1}{2} \int_{\mathcal{Q}_T} |\nabla B(u^\varepsilon)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t + \varepsilon \int_{\mathcal{Q}_T} B'(u^\varepsilon) |\nabla u^\varepsilon|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \\
&\leq \int_{\mathcal{Q}_T} \partial_t A(u^\varepsilon) \mathrm{d}\mathbf{x}\mathrm{d}t + \frac{1}{2} \int_{\mathcal{Q}_T} |\mathbf{f}(u^\varepsilon)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \leq C,
\end{aligned}$$

where C is a constant independent of ε. Thus (3.2) holds. To show the stated $\mathcal{DM}^2$ property, we use the same approach as in [21] (see also [6]). First define approximation sgn_η and $|\cdot|_\eta$ of the sign and modulus functions by

$$\mathrm{sgn}_\eta(\tau) := \begin{cases} 1, & \tau > \eta, \\ \frac{\tau}{\eta}, & |\tau| \leq \eta, \\ -1, & \tau < -\eta, \end{cases} \qquad |x|_\eta := \int_0^x \mathrm{sgn}_\eta(\zeta) \mathrm{d}\zeta, \qquad \eta > 0.$$

Multiply $(4.3)_1$ by $\mathrm{sgn}_\eta(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))\psi$ where $k \in \mathbb{R}$ and $\psi \in C^\infty(\overline{\mathcal{Q}}_T)$ and integrating by parts over $\mathcal{Q}_T$, using $(4.3)_3$, we get

$$\begin{aligned}
&\int_{\mathcal{Q}_T} (\nabla(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k)))^2 \mathrm{sgn}_\eta'(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))\psi \mathrm{d}x\mathrm{d}t \\
&= \int_{\mathcal{Q}_T} \{\mathbf{f}(u^\varepsilon) - \mathbf{f}(k) - \nabla(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))\}\mathrm{sgn}_\eta(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))\nabla\psi \mathrm{d}x\mathrm{d}t \\
&+ \int_{\mathcal{Q}_T} (\mathbf{f}(u^\varepsilon) - \mathbf{f}(k))\mathrm{sgn}_\eta'(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))\nabla(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))\psi \mathrm{d}x\mathrm{d}t \qquad (4.6)\\
&- \int_\Omega |u^\varepsilon - k|_\eta \psi \,|_0^T \,\mathrm{d}x + \int_{\mathcal{Q}_T} (u^\varepsilon - k)\mathrm{sgn}_\eta'(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))\partial_t(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))\psi \mathrm{d}x\mathrm{d}t \\
&+ \int_{\mathcal{Q}_T} (u^\varepsilon - k)\mathrm{sgn}_\eta(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))\partial_t\psi \mathrm{d}x\mathrm{d}t \\
&+ \int_0^T \int_{\partial\Omega} \mathbf{f}(k)\cdot\mathbf{n}\mathrm{sgn}_\eta(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))\psi \mathrm{d}\mathcal{H}^N \\
&=: I_\eta^1 + I_\eta^2 + \cdots + I_\eta^6.
\end{aligned}$$

Now we consider the limit of the righthand side of (4.6) for $\eta \to 0$. First we note that $\mathrm{sgn}(u^\varepsilon - k) = \mathrm{sgn}(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))$ due to the monotonicity of $B^\varepsilon(\cdot)$ and by using Lebesgue's theorem, we get as $\eta \to 0$,

$$I_\eta^1 \to \int_{\mathcal{Q}_T} \{\mathrm{sgn}(u^\varepsilon - k)(\mathbf{f}(u^\varepsilon) - \mathbf{f}(k)) - \nabla|B^\varepsilon(u^\varepsilon) - B^\varepsilon(k)|\}\nabla\psi \mathrm{d}x\mathrm{d}t.$$

To estimate I_η^2, we shall use the fact that $u\mathrm{sgn}_\eta'(u) \leq \chi_{\{u:0<|u|\leq\eta\}}$ (see [21]) and noting that $\sup_{|\xi|\leq M} B^{\varepsilon\prime}(\xi) \geq \varepsilon$, we get

$$\begin{aligned}
&|(\mathbf{f}(u^\varepsilon) - \mathbf{f}(k))\mathrm{sgn}_\eta'(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))\nabla(B^\varepsilon(u) - B^\varepsilon(k))| \\
&\qquad \leq \frac{\sup_{|\xi|\leq M} \mathbf{f}'(\xi)}{\varepsilon}|\nabla(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))|\chi_{\mathcal{I}(\varepsilon,\eta)}
\end{aligned}$$

where $\mathcal{I}(\varepsilon,\eta) = \{(x,t) : 0 \leq |B^\varepsilon(u^\varepsilon) - B^\varepsilon(k)| \leq \eta\}$. Therefore,

$$|I_\eta^2| \leq \frac{\sup_{|\xi|\leq M} \mathbf{f}'(\xi)}{\varepsilon}\|\psi\|_{L^\infty(\mathcal{Q}_T)} \int_{\mathcal{I}(\varepsilon,\eta)} |\nabla(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))|\mathrm{d}x\mathrm{d}t.$$

Observe that the meas $\mathcal{I}(\varepsilon,\eta) \to 0 \quad$ as $\quad \eta \to 0$, so $I_\eta^2 \to 0$ as $\eta \to 0$. Next, we see that as $\eta \to 0$,

$$I_\eta^3 \to I_0^3 := \int_\Omega |u^\varepsilon(x,T) - k|\psi(x,T) - |u_0^\varepsilon(x,T) - k|\psi(x,0)\mathrm{d}x$$

and thus $|I_0^3| \leq 2(M+k)\|\xi\|_{L^\infty(\mathcal{Q}_T)}$. The integrand of I_η^4 satisfying

$$\begin{aligned}
&|(u^\varepsilon - k)\mathrm{sgn}'_\eta(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))\partial_t(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))\psi| \\
&= |(u^\varepsilon - k)\mathrm{sgn}'_\eta(u^\varepsilon - k)\partial_t(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))\psi| \\
&\leq |\partial_t(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))\chi_{\{(x,t):0\leq|u^\varepsilon(x,t)-k|\leq\eta\}}\|\psi\|_{L^\infty(\mathcal{Q}_T)}.
\end{aligned}$$

The same argument employed for I_η^2 implies that $I_\eta^4 \to 0$ as $\eta \to 0$. Finally, using Lebesgue's theorem, we get

$$I_\eta^5 \to I_0^5 := \int_{\mathcal{Q}_T} |u^\varepsilon - k|\partial_t\psi \mathrm{d}x\mathrm{d}t$$

and

$$I_\eta^6 \to I_0^6 := \int_0^T \int_{\partial\Omega} \mathbf{f}(k)\cdot \mathbf{n}\mathrm{sgn}(B^\varepsilon(u^\varepsilon) - B^\varepsilon(u))\psi \mathrm{d}\mathcal{H}^N.$$

It's easy to see that $|I_0^6| \leq T|\partial\Omega||\mathbf{f}(k)|\|\psi\|_{L^\infty(\mathcal{Q}_T)}$. Collecting the estimates on I_η^1 to I_η^6 yields that all terms of the righthand side of (4.6) possess a limit as $\eta \to 0$ and are in particular uniformly bounded with respect to η. Therefore, taking $\psi \equiv 1$, we see that there exists a constant C_1, depending possibly on ε (but not on η), such that

$$\int_{\mathcal{Q}_T} (\nabla B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))^2 \mathrm{sgn}'_\eta(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))\mathrm{d}x\mathrm{d}t \leq C_1(\varepsilon).$$

Consequently, the sequence

$$\{E_{\varepsilon,\eta}\}_{\eta>0} := \{(\nabla B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))^2\mathrm{sgn}'_\eta(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))\}_{\eta>0}$$

is bounded in $L^1(\mathcal{Q}_T)$ with respect to η and therefore also in $\mathcal{C}(\overline{\mathcal{Q}}_T)'$, the dual space of $\mathcal{C}(\overline{\mathcal{Q}}_T)$ of continuous functions on $\overline{\mathcal{Q}}_T$. By compactness of the weak-$\star$ topology of $\mathcal{C}(\overline{\mathcal{Q}}_T)'$, we deduce that, up to subsequence, the sequence $\{E_{\varepsilon,\eta}\}$ converges towards an element $E_\varepsilon \in \mathcal{C}(\overline{\mathcal{Q}}_T)'$ in the weak-$\star$ topology. Thus for any $\psi \in \mathcal{C}^\infty(\overline{\mathcal{Q}}_T)$ we can pass to the limit $\eta \to 0$ in (4.6) to get

$$\begin{aligned}
< E_\varepsilon, \psi > \quad &= \int_{\mathcal{Q}_T} \{\mathrm{sgn}(u^\varepsilon - k)(\mathbf{f}(u^\varepsilon) - f(k)) - \nabla|B^\varepsilon(u^\varepsilon) - B^\varepsilon(k)|\}\nabla\psi \mathrm{d}x\mathrm{d}t \\
&- \int_\Omega \{|u^\varepsilon(x,T) - k|\psi(x,T) - |u^\varepsilon(x,0) - k|\psi(x,0)\}\mathrm{d}x \qquad (4.7)\\
&+ \int_{\mathcal{Q}_T} |u^\varepsilon - k|\partial_t\psi + \int_0^T\int_{\partial\Omega} \mathbf{f}(k)\cdot n\mathrm{sgn}_\eta(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))\psi \mathrm{d}\mathcal{H}^N.
\end{aligned}$$

On the other hand, since $\mathrm{sgn}'_\eta \geq 0$, we have $E_{\varepsilon,\eta} > 0$ for every $\varepsilon, \eta > 0$. Therefore, we get

$$\begin{aligned}
&\frac{|<E_\varepsilon,\psi>|}{\|\psi\|_{L^\infty(\mathcal{Q}_T)}}\\
&= \lim_{\eta\to 0}\frac{1}{\|\psi\|_{L^\infty(\mathcal{Q}_T)}}\int_{\mathcal{Q}_T}(\nabla B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))^2 \mathrm{sgn}'_\eta(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))\psi \mathrm{d}x\mathrm{d}t\\
&\leq \limsup_{\eta\to 0}\int_{\mathcal{Q}_T}(\nabla B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))^2 \mathrm{sgn}'_\eta(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))\mathrm{d}x\mathrm{d}t
\end{aligned}$$

Thus, we get from (4.7) with $\psi \equiv 1$

$$\begin{aligned}
\frac{<E_\varepsilon,\psi>}{\|\psi\|_{L^\infty}} \quad &\leq -\int_\Omega \{|u^\varepsilon(x,T) - k| - |u^\varepsilon(x,0) - k|\}\mathrm{d}x\\
&+\int_0^T\int_{\partial\Omega}\mathbf{f}(k)\cdot\mathbf{n}\mathrm{sgn}_\eta(B^\varepsilon(u^\varepsilon) - B^\varepsilon(k))\psi \mathrm{d}\mathcal{H}^N \qquad (4.8)
\end{aligned}$$

Using estimate from Lemma (4.1), we deduce that there exists a constant C_2 which does not depend on ε such that $|<E_\varepsilon,\psi>| \leq C_2\|\psi\|_{L^\infty(\mathcal{Q}_T)}$ for all $\varepsilon > 0$. Consequently, E_ε is bounded in $\mathcal{C}(\overline{\mathcal{Q}}_T)'$ and up to subsequence E_ε converges in the weak-$\star$ topology to a function $E \in \mathcal{C}(\overline{\mathcal{Q}}_T)'$. Now we pass to the limit $\varepsilon \to 0$ in (4.7). Note that $|u^\varepsilon - k|$ converges strongly to $|u-k|$ in $\mathcal{C}(0,T;L^1(\Omega))$ and $\mathrm{sgn}(u^\varepsilon - k)(\mathbf{f}(u^\varepsilon) - \mathbf{f}(k) - \nabla B^\varepsilon(u^\varepsilon))$ converges weakly in $L^2(\mathcal{Q}_T)$ to $\mathrm{sgn}(u-k)(\mathbf{f}(u) - \mathbf{f}(k) - \nabla B(u))$. Thus we conclude that for all $\phi \in \mathcal{C}_0^\infty(\mathcal{Q}_T)$,

$$<E,\phi> = \int_{\mathcal{Q}_T}\{|u-k|\partial_T\phi + \mathrm{sgn}(u-k)(\mathbf{f}(u) - \mathbf{f}(k) - \nabla B(u))\cdot\nabla\phi\}\mathrm{d}x\mathrm{d}t \quad (4.9)$$

Since E is a Radon measure, we obtain from (4.9) that for all $\phi \in \mathcal{C}_0^\infty(\mathcal{Q}_T)$,

$$\int_{\mathcal{Q}_T}\{|u-k|\partial_t\phi + \mathrm{sgn}(u-k)(\mathbf{f}(u) - \mathbf{f}(k) - \nabla B(u))\cdot\nabla\phi\}\mathrm{d}x\mathrm{d}t \leq C\|\phi\|_{L^\infty(\mathcal{Q}_T)}. \quad (4.10)$$

This in particular implies the $\mathcal{DM}^2$ property (3.2). □

The existence of entropy solutions is presented in the following theorem.

Theorem 4.1. *Suppose $u_0(\mathbf{x}) \in [0,M]$ for a.e. $\mathbf{x} \in \Omega$ and Assumption 2.1 and 2.2 hold. Moreover, we require that assumptions* (4.1) *and* (4.2) *are satisfied. Then the limit function $u(x,t)$ in Lemma 4.1 is an entropy solution of initial-boundary value problem* (1.1).

Proof. We divide our proof into two steps.
Step1. Let $\eta \in \mathcal{C}^2(\mathbb{R})$ be a convex function and $\phi(\mathbf{x},t) \in \mathcal{C}_0^\infty([0,T)\times\Omega)$. Multiplying $(4.3)_1$ by $\eta'(u^\varepsilon)\phi(\mathbf{x},t)$, we get

$$\int_{\mathcal{Q}_T}\eta(u^\varepsilon)\phi_t\mathrm{d}\mathbf{x}\mathrm{d}t + \int_{\mathcal{Q}_T}(\mathbf{q}(u^\varepsilon) - \nabla p(u^\varepsilon) + \varepsilon\nabla\eta(u^\varepsilon))\cdot\nabla\phi\mathrm{d}\mathbf{x}\mathrm{d}t + \int_\Omega \eta(u_0^\varepsilon)\phi(0,x)\mathrm{d}\mathbf{x} \geq 0, \quad (4.11)$$

where $\mathbf{q}' = \eta'\mathbf{f}'$ and $p' = \eta' B'$. Let $\varepsilon \to 0$, we get

$$\int_{\mathcal{Q}_T} \eta(u)\phi_t \mathrm{d}\mathbf{x}\mathrm{d}t + \int_{\mathcal{Q}_T} (\mathbf{q}(u) - \nabla p(u)) \cdot \nabla\phi \mathrm{d}\mathbf{x}\mathrm{d}t + \int_{\Omega} \eta(u_0)\phi(0,x)\mathrm{d}\mathbf{x} \geq 0. \quad (4.12)$$

By approximation, the above inequality still holds for $\eta(u) = |u - k|$ for any $k \in \mathbb{R}$. So, conditions (2) and (3) hold.

Step 2. Let $\varphi(\mathbf{x}, t) \in \mathcal{C}_0^\infty(\overline{\mathcal{Q}}_T)$. Multiplying $(4.3)_1$ by $\varphi(\mathbf{x}, t)$ and $\mathrm{sgn}(u^\varepsilon - k)\varphi(\mathbf{x}, t)$ respectively and integrating over $\mathcal{Q}_T$, using $(4.3)_3$ and (4.1), we get

$$\int_{\mathcal{Q}_T} u^\varepsilon \partial_t\varphi \mathrm{d}\mathbf{x}\mathrm{d}t + \int_{\mathcal{Q}_T} \{\mathbf{f}(u^\varepsilon) - \mathbf{f}(k) - \nabla B(u^\varepsilon) - \varepsilon\nabla u^\varepsilon\} \cdot \nabla\varphi \mathrm{d}\mathbf{x}\mathrm{d}t = 0, \quad (4.13)$$

and for $k \geq u_c$

$$\int_{\mathcal{Q}_T} |u^\varepsilon - k|\partial_t\varphi \mathrm{d}\mathbf{x}\mathrm{d}t + \int_{\mathcal{Q}_T} \mathrm{sgn}(u^\varepsilon - k)\{\mathbf{f}(u^\varepsilon) - \mathbf{f}(k) - \nabla B(u^\varepsilon) - \varepsilon\nabla u^\varepsilon\} \cdot \nabla\varphi \mathrm{d}\mathbf{x}\mathrm{d}t \geq 0. \quad (4.14)$$

Let $\varepsilon \to 0$, we have

$$\int_{\mathcal{Q}_T} u\partial_t\varphi \mathrm{d}\mathbf{x}\mathrm{d}t + \int_{\mathcal{Q}_T} \{\mathbf{f}(u) - \mathbf{f}(k) - \nabla B(u)\}\nabla\varphi \mathrm{d}\mathbf{x}\mathrm{d}t = 0, \quad (4.15)$$

and for $k \geq u_c$

$$\int_{\mathcal{Q}_T} |u - k|\partial_t\varphi \mathrm{d}\mathbf{x}\mathrm{d}t + \int_{\mathcal{Q}_T} \mathrm{sgn}(u - k)\{\mathbf{f}(u) - \mathbf{f}(k) - \nabla B(u)\} \cdot \nabla\varphi \mathrm{d}\mathbf{x}\mathrm{d}t \geq 0. \quad (4.16)$$

Take $\varphi(\mathbf{x}, t) = (1 - \zeta_\delta(\mathbf{x}))\psi(\mathbf{x}, t)$ in (4.15) and (4.16) with $\psi(\mathbf{x}, t) \in \mathcal{C}_0^\infty(\overline{\mathcal{Q}}_T)$ and then let $\delta \to 0$, we have

$$\lim_{\delta\to 0} \int_{\mathcal{Q}_T} (\mathbf{f}(u) - \nabla B(u)) \cdot \nabla\zeta_\delta\psi \mathrm{d}\mathbf{x}\mathrm{d}t = 0, \quad (4.17)$$

and for $k \geq u_c$

$$\lim_{\delta\to 0} \int_{\mathcal{Q}_T} \mathrm{sgn}(u - k)(\nabla B(u) - \mathbf{f}(u)) \cdot \nabla\zeta_\delta\psi \mathrm{d}\mathbf{x}\mathrm{d}t \geq 0. \quad (4.18)$$

Next we shall show that (4.18) holds for $k < u_c$. Using (4.1), (4.2) and (4.17), we have for $k < u_c$

$$\begin{aligned}
&\lim_{\delta\to 0} \int_{\{u \geq u_c\}\cup\{u<u_c\}} \mathrm{sgn}(u - k)(\mathbf{f}(u) - \nabla B(u)) \cdot \nabla\zeta_\delta\psi \\
&= \lim_{\delta\to 0} \int_{\{u\geq u_c\}} (\mathbf{f}(u) - \nabla B(u)) \cdot \nabla\zeta_\delta\psi + \lim_{\delta\to 0} \int_{\{u<u_c\}} \mathrm{sgn}(u - k)\mathbf{f}(u) \cdot \nabla\zeta_\delta\psi \\
&= \lim_{\delta\to 0} \int_{\{u<u_c\}} (\mathrm{sgn}(u - k) - 1)\mathbf{f}(u) \cdot \nabla\zeta_\delta\psi = 0.
\end{aligned}$$

The last equality in the above formula holds since $\nabla B(u) = 0$ on the set $\{u < u_c\}$ and thus $\mathbf{f}(u^\tau) \cdot n = 0$ a.e. on the set $\partial\{u < u_c\} \cap \partial\Omega$ by (4.17). Combining

the regularity results from Lemma 4.2, the existence of entropy solutions follows immediately. □

References

[1] B. P. Andreianov, F. Bouhsiss, Uniqueness for an ellipitic-parabolic problem with Neumann boundary condition, *J. Evo. Equ* **4** (2004) 273-295.

[2] G. Anzellotti, Pairings between measures and bounded functions and compensated compactness, *Ann. Mat. Pura Appl.*, **135** (1983), 293-318.

[3] J. R. Anderson, Local existece and uniqueness of solutions of degenerate parabolic equations, *Comm. Partial Diff. Equa.* **16** (1991), 105-143.

[4] J. Bear, Dynamics of Fluids in Porous Media. Elsevier, New York, NY, 1972.

[5] R. Bürger, H. Frid, K.H. Karlsen, On the well-posedness of entropy solutions to conservation laws with a zero-flux boundary condition, *J. Math. Anal. Appl.* **326** (2007), 108-120.

[6] R. Bürger, H. Frid, K.H. Karlsen, On a free boundary problem for a strongly degenerate quasi-linear parabolic equation with an application to a model of pressure filtration, *SIAM J. Math. Anal.* **34** (2003), 611-635.

[7] R. Bürger, S. Evje, K. H. Karlsen, On strongly degenerate convection-diffusion problems modeling sedimentation-consolidation processes, *J. Math. Anal. Appl.* **247** (2000), 517-556.

[8] M. C. Bustos, F. Comcha, R. Burge, E. M. Tory, Sedimentation and Thicking: Phenomenological Foudation and Mathematical Theory, Kluwer Academic Publishers: Dordrecht, Netherlands, 1999.

[9] J. Carillo, Entropy solutions for nonlinear degenerate problems, *Arch. Ration. Mech. Anal.* **147** (1999), 269-361.

[10] G.Q. Chen, H. Frid, Divergence-measure fields and hyperbolic conservation laws, *Arch. Ration. Mech. Anal.* **147** (1999), 89-118.

[11] G.Q. Chen, H. Frid, Extended divergence-measure fields and the Euler equations for gas dynamics, *Comm. Math. Phys.* **236** (2003), 251-280.

[12] G.Q. Chen, B. Perthame, Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations, *Ann. I. H. Poincaré*, **20** (2003), 645-668.

[13] F. Golse, P.L. Lions,B. Perthame, R. Sentis, Regularity of the moments of the solution of a transport equation, *J. Funct. Anal.* **76** (1988), 110-125.

[14] Y. Hu, Y. C. Li, On the zero flux boundary problem for conservation laws with source terms, prepint.

[15] L.V. Juan, The porous medium equation: mathematical theory, Oxford University Press, 2006.

[16] K. Kobayasi, H. Ohwa, Uniqueness and existence for anisotropic degenerate parabolic equations with boundary conditions on a bounded rectangle, *J. Differential Equations* **252** (2012), 137-167.

[17] Y.S. Kwon, Strong traces for degenerate parabolic-hyperbolic equations, *Discrete Contin. Dyn. Syst.* **25** (2009), 1275-1286.

[18] O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva, *Linear and Quasi-linear Equations of Parabolic type.* Amer. Math. Soc. Providence, RI,1968

[19] Y.C. Li, Q. Wang, Homogeneous Dirichlet Problems for Quasilinear Anisotropic Degenerate Parabolic-Hyperbolic Equations, *J. Differential Equations* **252** (2012), 4719-4741.

[20] P.L. Lions, B. Perthame, E. Tadmor, A kinetic formulation of multidimensional scalar conservation laws and related equations, *J. Amer. Math. Soc.* **7** (1994), 169-191.

[21] C. Mascia, A. Poretta, A. Terracina, Nonhomogeneous Dirichlet problems for degenetate parabolic-hyperbolic equations, *Arch. Ration. Mech. Anal.* **163** (2002), 87-124.

[22] A. Michel, J. Vovelle, Entropy formulation for parabolic degenerate equations with general Dirichlet boundary conditions and application to the convergence of FV methods. *SIAM J. Numer. Anal.* **41** (2003), 2262-2293.

[23] L. Tartar, *Compensated compactness and applications to partial differential equations*, In Research Notes in Mathematics, 39, Nonlinear Analysis and Mechanics: Heriot-Watt Symposium, Vol. 4, ed. R.J. Knops, Pitman Press: Boston-London, 1979.

[24] A. Vasseur, Strong traces for solutions of multidimensional scalar conservation laws, *Arch. Ration. Mech. Anal.* **160** (2001), 181-193

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